Tutorial on computational neuroscience

Course G: Basic neurosciences 2
Emil Ratko-Dehnert
summer term 2012

NCP
Neuro-Cognitive Psychology

Tutorial aims

- Basic mathematical concepts relevant for computational neuroscience
- Hands-on experience with simbrain (neural networks simulator)

Organisational Infos

Room: small mensa CIP-Pool

Dates: 23.05. Basic Maths for Comp. Neuros.

30.05., 06.06., 13.06. simbrain labs 1- 6

Time : 10:00 – 12:00 (s.t.)

Materials: via dropbox and/or on my homepage

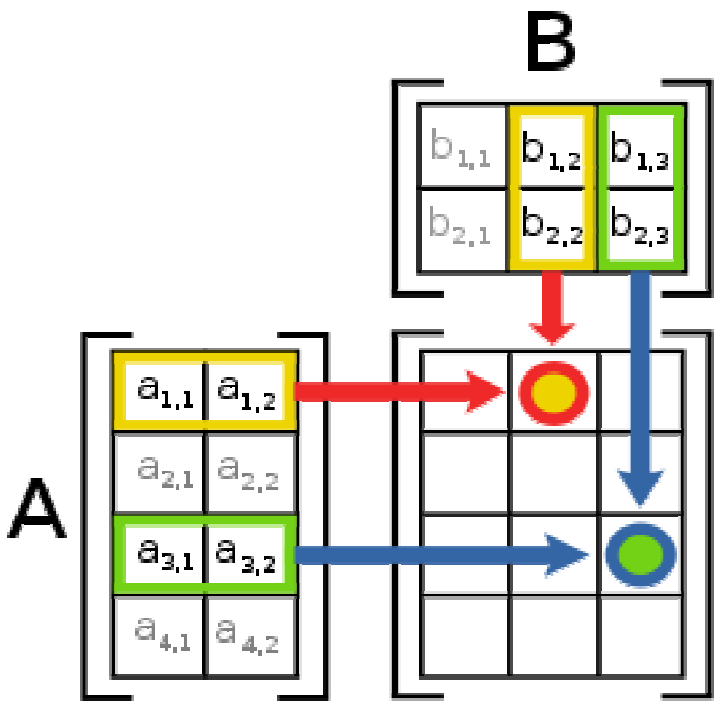
Grading: math sheet + active lab participation

Literature

1. Essential Mathematics for Neuroscience *(Fabian Sinz)*

2. Mathematics for Computational Neuroscience & Imaging *(John Porrill)*

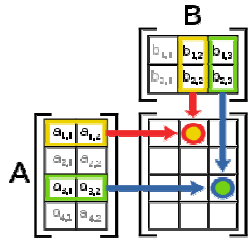
3. Simbrain: A visual framework for neural network analysis and education *(Jeff Yoshimi)*



Session 1.1

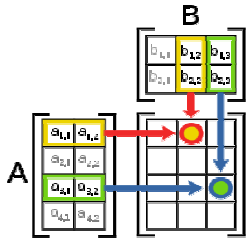
MATRIX CALCULUS

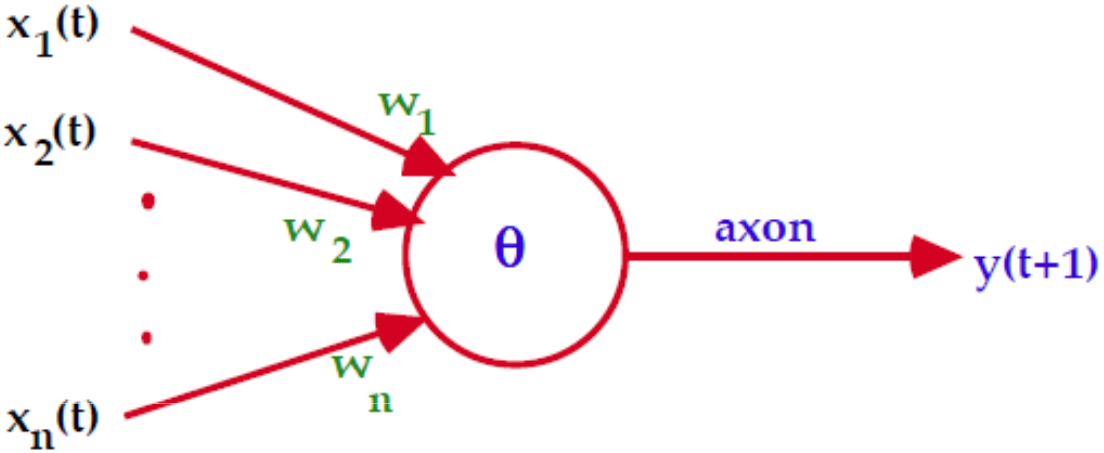
Matrices relevant for...



- *Algebra*: Solving systems linear equations (**Ax = b**)
- *Statistics*: LLS, covariance matrices of random variables
- *Calculus*: differentiation of multidimensional functions
- *Physics*: mechanics, linear combinations of quantum states and many more
- *Here*: Modelling neural dynamics (e.g. propagation of activity across a neural net)

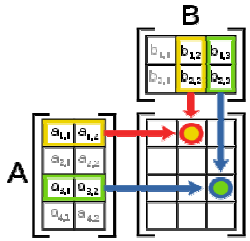
McCulloch-Pitts neuron

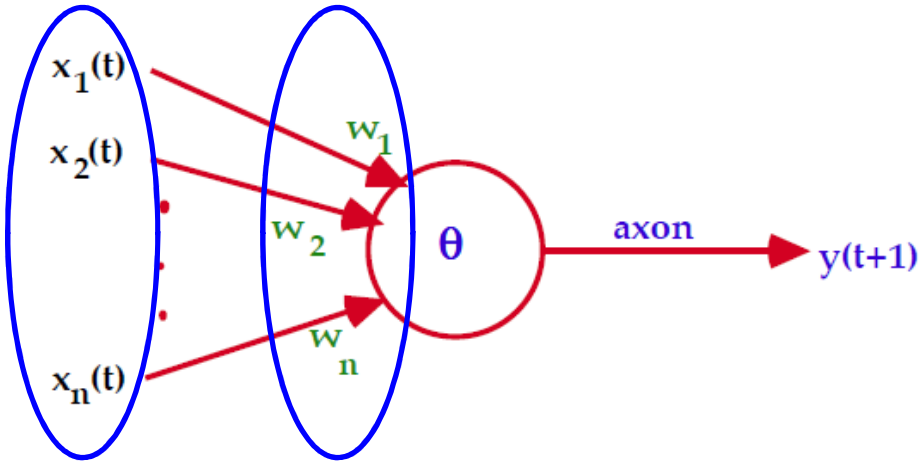




$$y(t+1) = \begin{cases} 1, & if \ \sum_i w_i x_i(t) \geq \theta \\ 0, & if \ \sum_i w_i x_i(t) < \theta \end{cases}$$

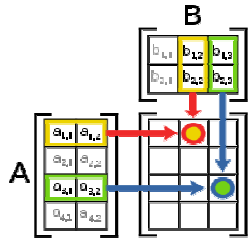
Inner product of vectors



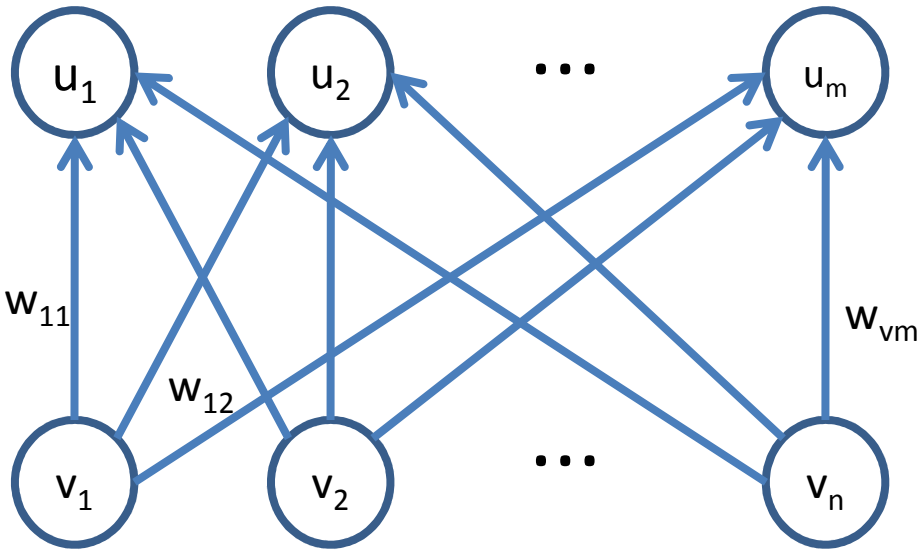


$$\langle x, w \rangle = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_i \\ \vdots \\ w_n \end{pmatrix} = \sum_i w_i x_i$$

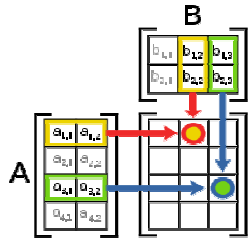
Linear associator



- n*m inner products(!)
- Alternative: Use matrix calculus ...

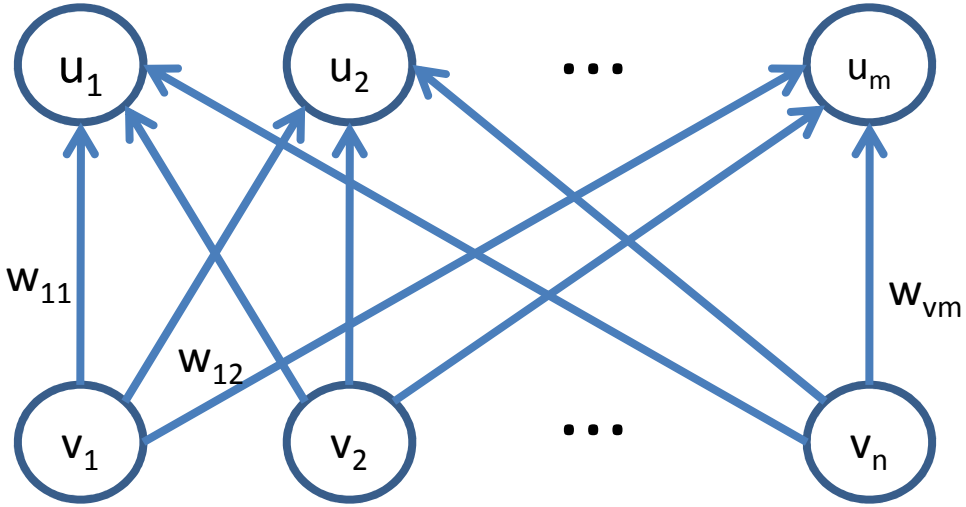


Matrix interpretation



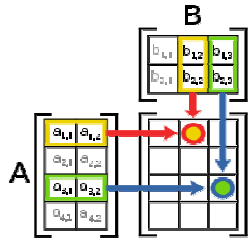
- Weights = coeff. matrix W
- Inputs/ Outputs = vectors

  v, u



$$u = W \cdot v$$

What is a matrix?



- Def: A matrix $\boldsymbol{A = (a_{i,j})}$ is an array of numbers or variables

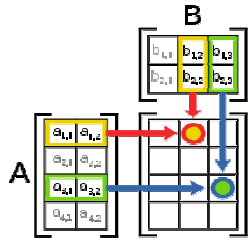
- It has ***m*** rows and ***n*** columns (***dim = (m, n)***)

m

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & & & & \vdots \\ \vdots & & & & & \vdots \\ a_{m,1} & \cdots & \cdots & \cdots & \cdots & a_{m,n} \end{bmatrix}$$

n

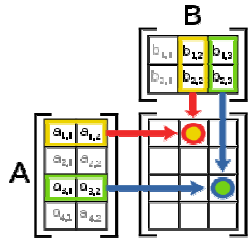
Connection to vectors



- Trivially, every vector can be interpreted as a matrix
- e.g. ***x*** is an ***(n, 1)***-matrix
- Thus all the following statements hold true for them as well

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix}$$

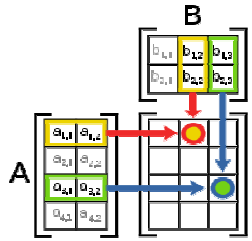
Equality of matrices



- Two matrices are equal, if and only if (iff) the have the same dimension ***(m, n)*** and all the elements are identical

$$A = B \iff A_{i,j} = B_{i,j}, \ \forall i, j$$

How are matrices added up?



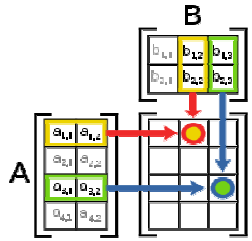
- Addition of two ***(2, 2)***-matrices ***A, B*** performed component-wise:

$$\begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 1 & -1 \end{bmatrix}$$

A **B** **A+B**

- Note that „***+***“ is commutative, i.e. ***A+B = B+A***

Scalar Multiplication

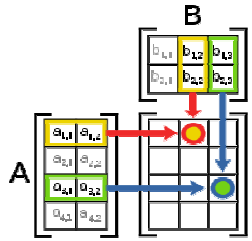


- Scalar Multiplication of a **(2, 2)**-matrix ***A*** with a scalar ***c***

$$\underset{\mathbf{c}}{2}\cdot\underset{\mathbf{A}}{\begin{bmatrix}1 & 4\\ 0 & -2\end{bmatrix}}=\underset{\mathbf{cA}}{\begin{bmatrix}2 & 8\\ 0 & -4\end{bmatrix}}$$

- Again commutativity, i.e. ***c*A = A*c***

Matrix multiplication

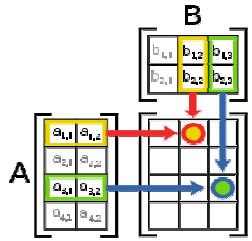


- Matrix multiplication of matrices ***C (2-by-3)*** and ***D (3-by-2)*** to ***E (2-by-2)***:

$$\underset{\mathbf{C}}{\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix}} \times \underset{\mathbf{D}}{\begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}} = \underset{\mathbf{E}}{\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}}$$

$$E_{11} = 1 \cdot 3 + 0 \cdot 2 + 2 \cdot 1 = 5$$

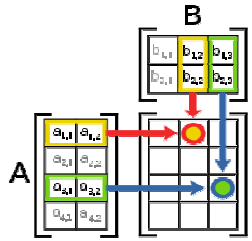
Falk-Schema



$$A \cdot B = C$$

$$\begin{array}{ccc|ccc}
 & & & b_{11} & b_{12} & b_{13} \\
 & & & b_{21} & b_{22} & b_{23} \\
 & & & b_{31} & b_{32} & b_{33} \\
\hline
a_{11} & a_{12} & a_{13} & c_{11} & c_{12} & c_{13} \\
a_{21} & a_{22} & a_{23} & c_{21} & c_{22} & c_{23} \\
a_{31} & a_{32} & a_{33} & c_{31} & c_{32} & c_{33}
\end{array}$$

Matrix specifics

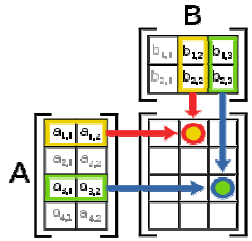


!Warning!

One can only multiply matrices if their **dimensions** correspond, i.e. **(m,n) * (n, k) → (m, k)**

- And generally: if ***A*B*** exists, ***B*A*** need not
- Furthermore: if ***A*B***, ***B*A*** exists, they need not be equal!

Transposition

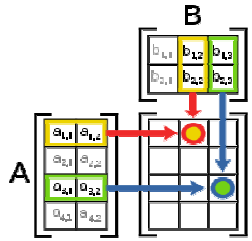


- Transposition of a **2-by-3** matrix ***A*** → ***A^T***

$$\underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & -6 & 9 \end{bmatrix}}_{\mathbf{A}}^{T} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & -6 \\ 4 & 9 \end{bmatrix}}_{\mathbf{A}^{\mathbf{T}}}$$

- It holds, that $A^{TT} = A$.

Matrix inversion



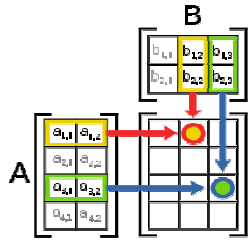
Square case, i.e. **dim = (n, n)**

- If **A** is regular (determinant is non-zero), then $\mathbf{A}^{-1}$ exists, with $\mathbf{A} * \mathbf{A}^{-1} = \mathbf{A}^{-1} * \mathbf{A} = \mathbf{I}_n$

Non-square matrices (**dim = (n, m)**)

- **A** with **dim = (n, m)** has a right inverse **B** with **dim (m, n)**, if the rank of **A** is **n** and a left inverse **B‘** with **dim (n, m),** if the rank of **A** is **m.**
- It holds that $\mathbf{A} * \mathbf{B} = \mathbf{I}_n$ and $\mathbf{B}' * \mathbf{A} = \mathbf{I}_m$

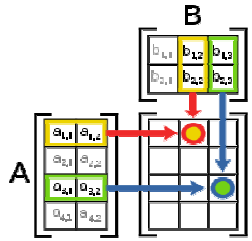
Methods of inversion



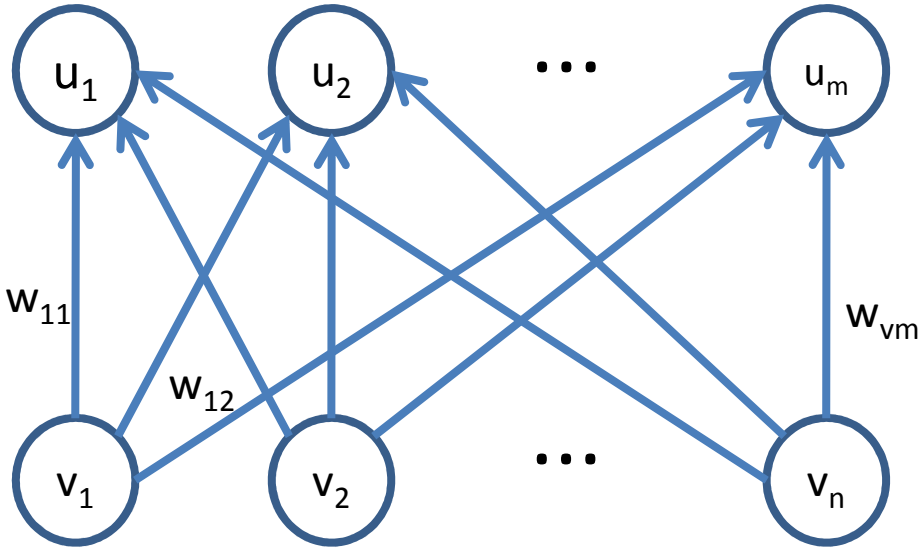
- Gauss-Jordan elimination algorithm
- Cramer's Rule
- For **dim = (2, 2)** there exists a short and explicit formula

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \underbrace{\frac{1}{ad-bc}}_{1/\det(A)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

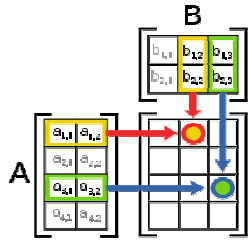
Back to linear associators



$$u = W \cdot v$$

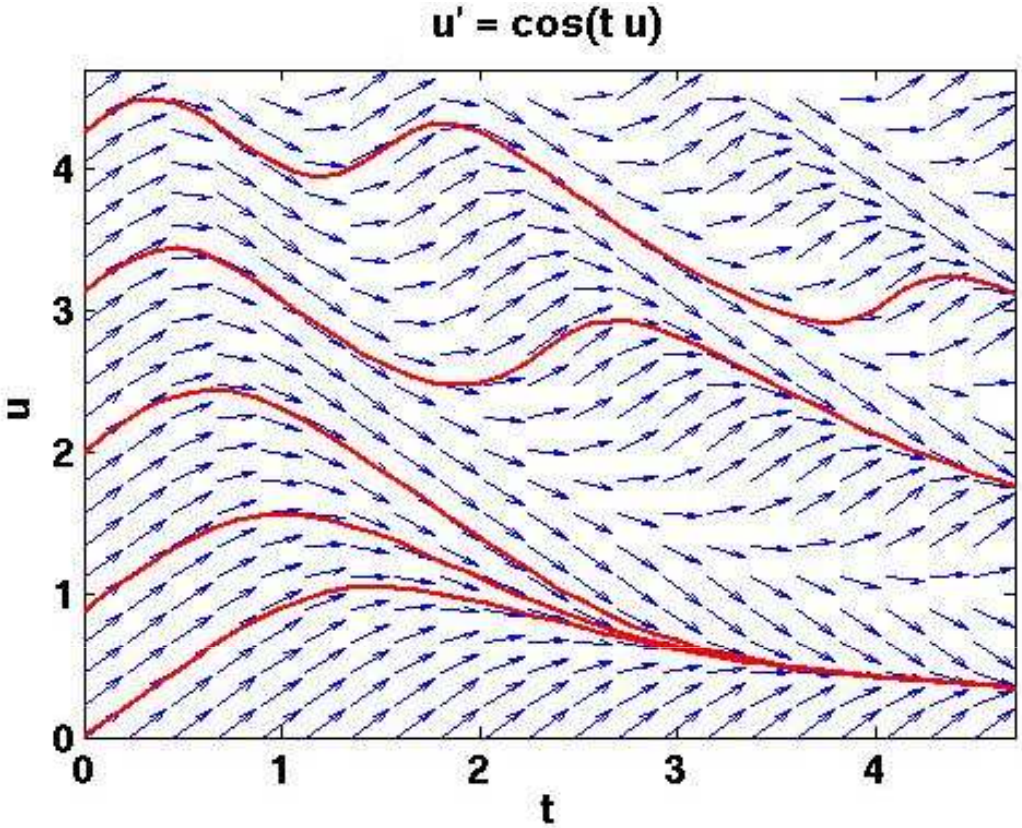


Back to linear associators



$$u = W \cdot v$$

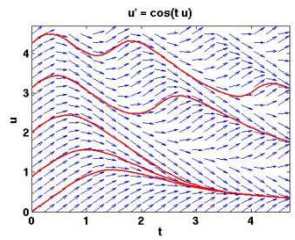
- We need different operations to address issues like
  - What will be the output **u** of a given input **v**, when we know the configuration of the weights **W**? -->
  - For a given output **u** and weight matrix **W**, what was the input **v**? -->
  - Compute the weight matrix **W** for desired input/output associations **v/u**. -->



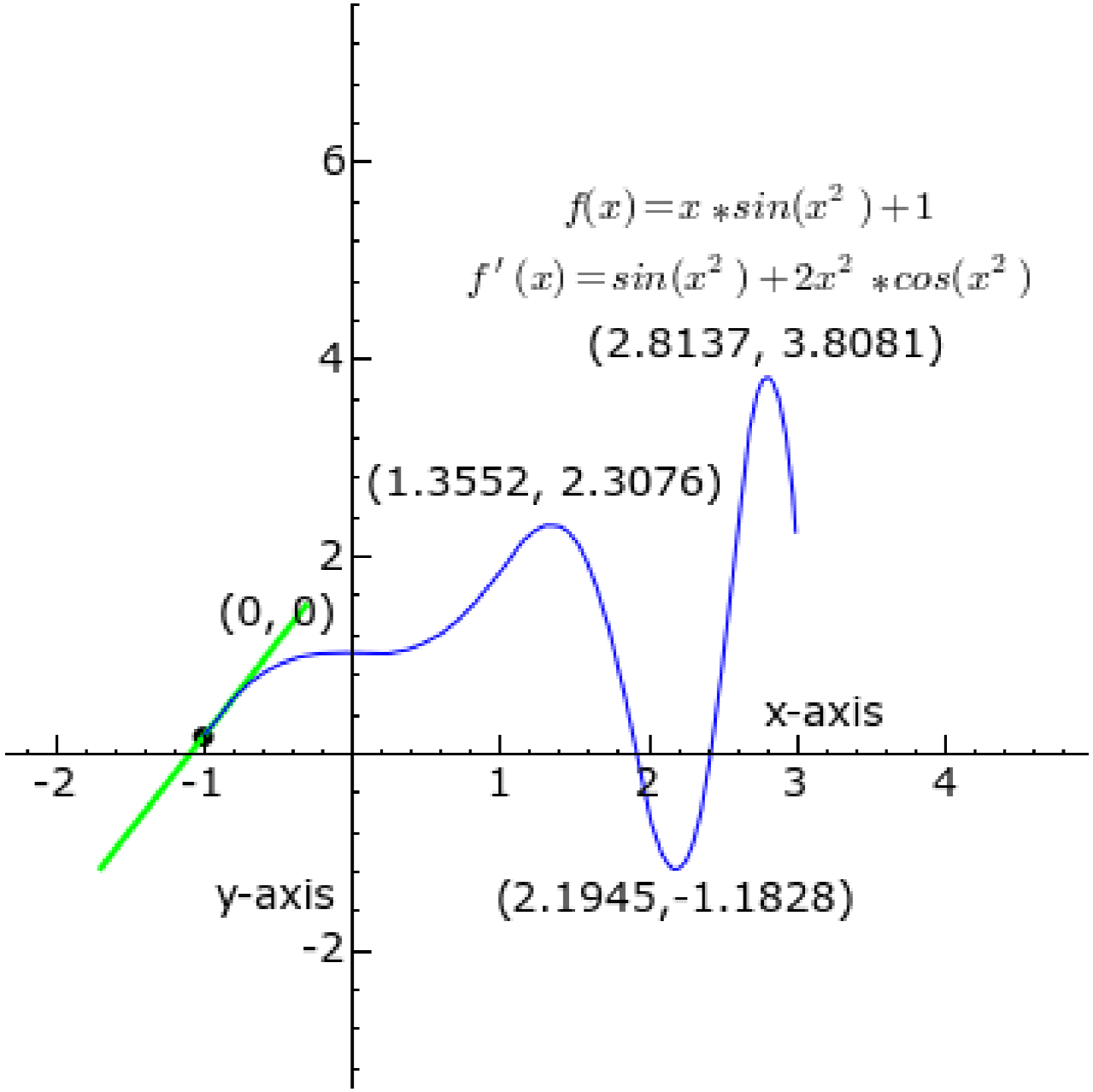
Session 1.2

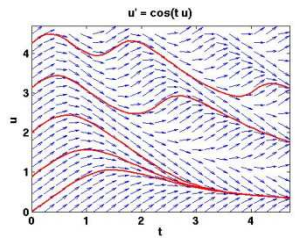
DIFFERENTIAL EQUATIONS

In calculus...

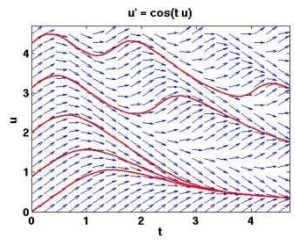


- one is generally interested in
  - solutions of functions
  - derivatives of functions
  - min/ max points
  - limits of series/ convergence
  - integrals of functions, etc.


$f(x) = x * sin(x^2) + 1$
$f'(x) = sin(x^2) + 2x^2 * cos(x^2)$
(2.8137, 3.8081)
(1.3552, 2.3076)
(0, 0)
x-axis
y-axis
(2.1945,-1.1828)
6
4
2
-2
-2
-1
1
2
3
4


u' = cos(t u)
u
t

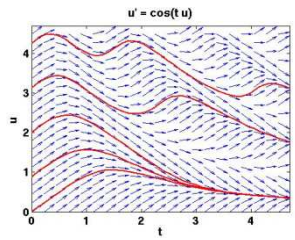
Differential Equation



- is a mathematical equation for an unknown function **u(x)** that relates the values of **u(x)** and its derivatives **u'(x), u''(x),** …

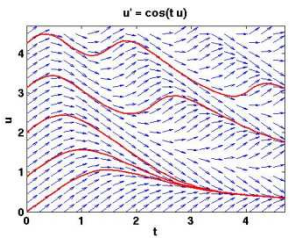
$$\frac{du}{dx} = cu + x^2 \Rightarrow u(x)?$$

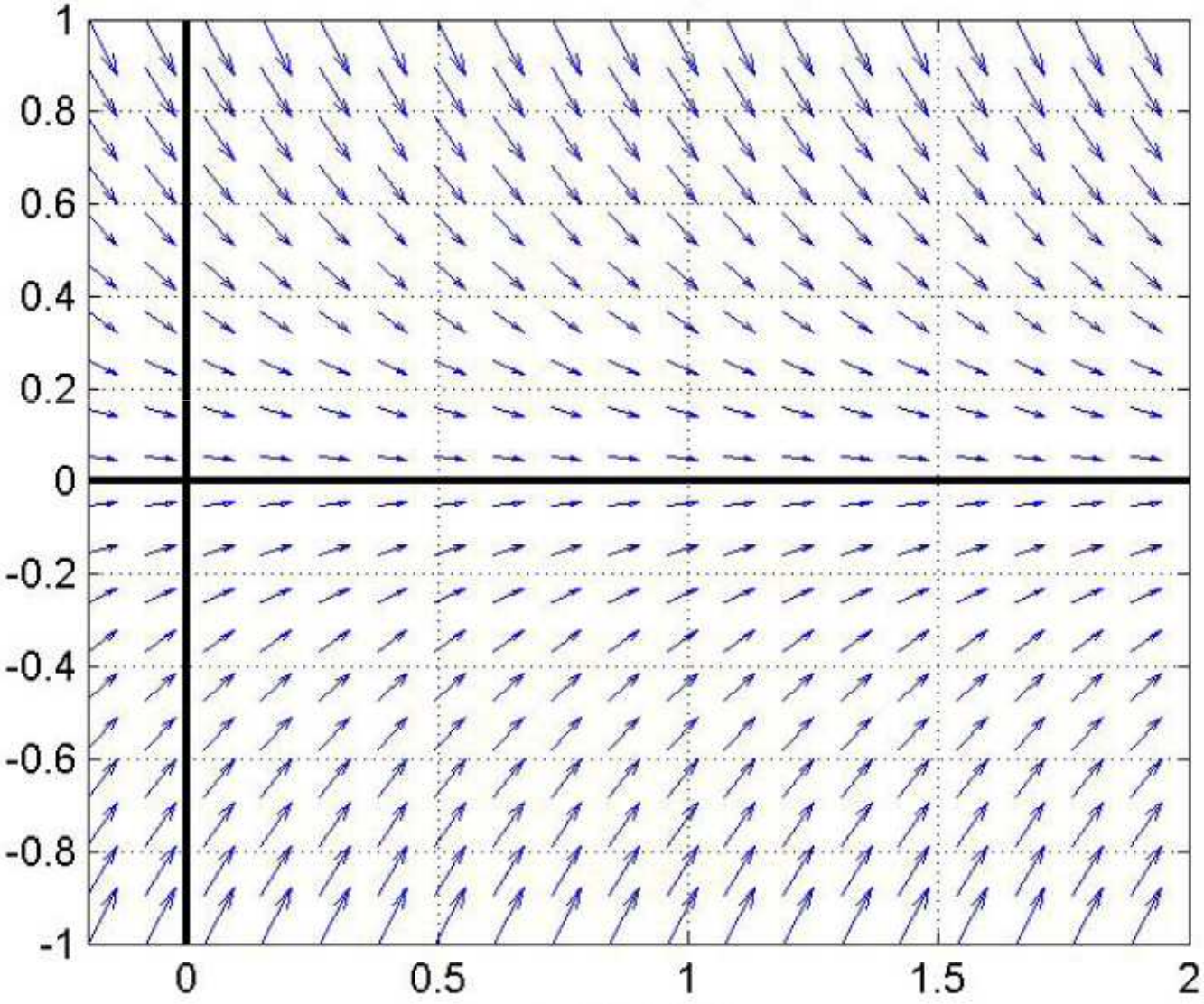
Applications



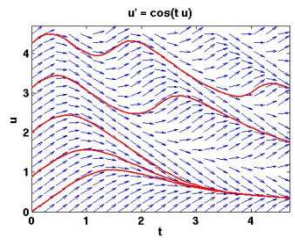
- Which function‘s slope matches the direction of the flow field (defined by the ODE) at every point?
- Has furthered most parts of physics
  - classical mechanics
  - statistical mechanics
  - dynamical systems
- But also applied fields like financial, biological and neurosciences

Applications

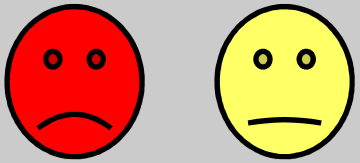


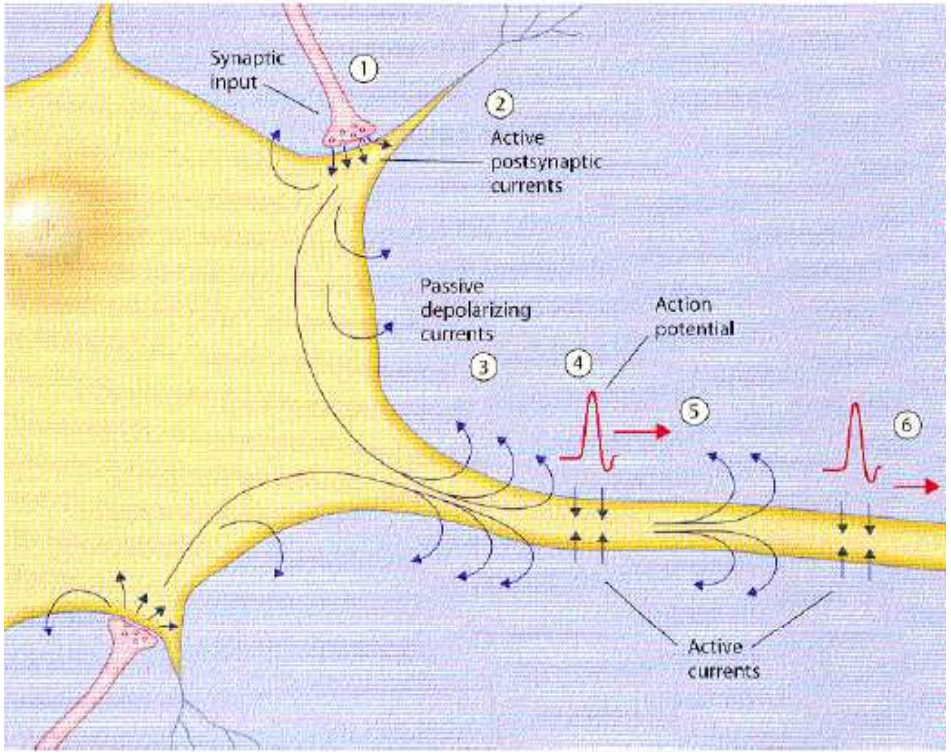


Classification



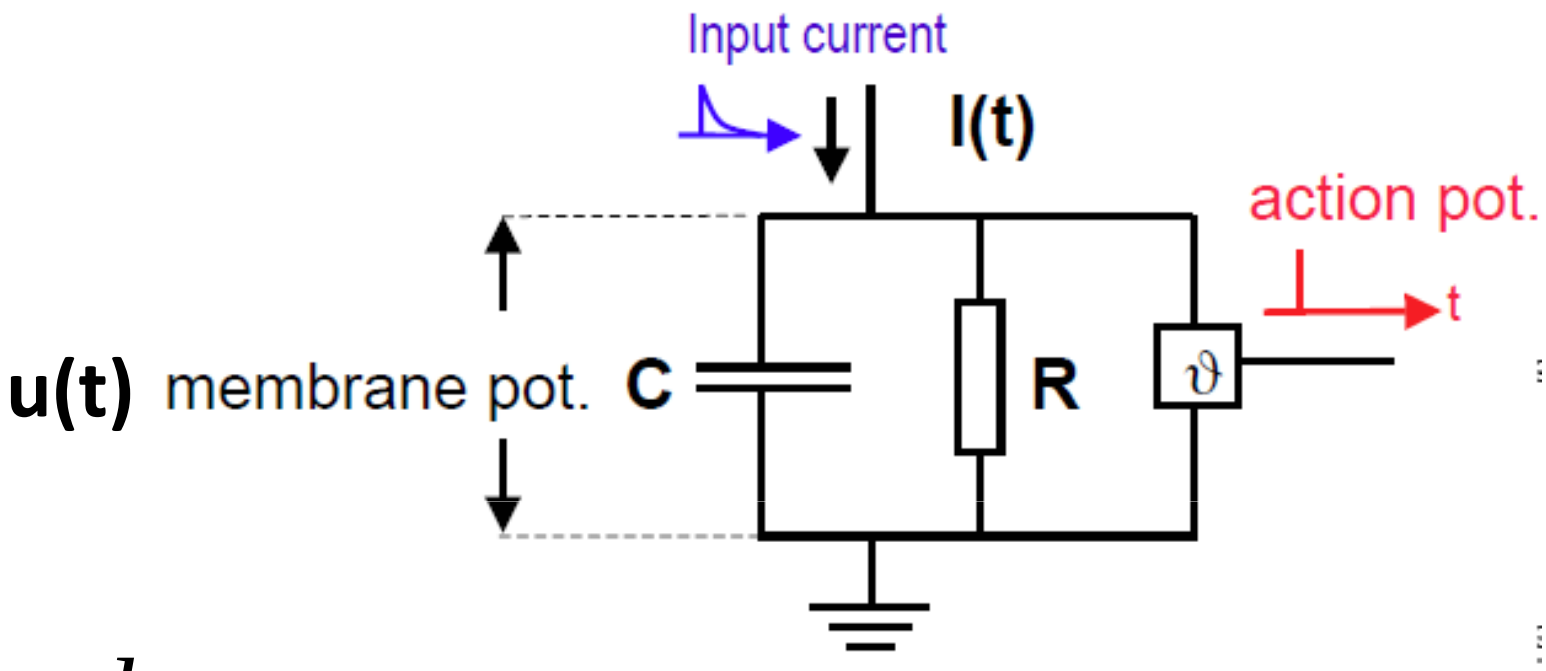
- Ordinary Differential Equations (ODEs)
  - Growth processes; Newtons 2nd law of motion
- Partial Differential Equations (PDEs)
  - Heat Equation, Wave Equation
- Stochastic Differential Equations (SDEs)
  - Black-Scholes Formula

	ODEs	PDEs	SDEs
Solutions?			
Unique?			
Analytic?/ Explicit			
Stability/ Linearity?			



THE HODGKIN-HUXLEY MODEL

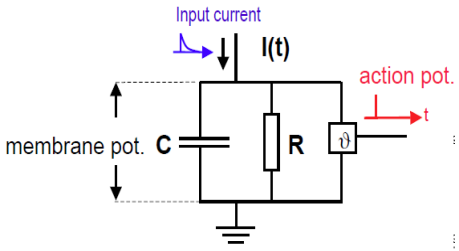
Hodgkin-Huxley's model of a neuron



$$\frac{du}{dt} = -k \cdot u(t) + m \cdot I(t)$$

$$\frac{du}{dt} = -\frac{1}{RC} \cdot u(t) + \frac{1}{C} \cdot I(t)$$

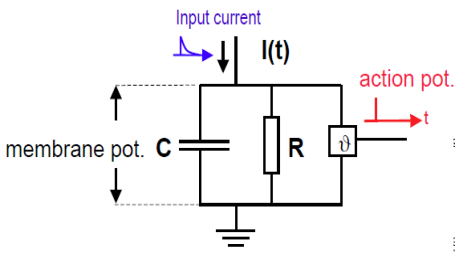
Deriving the model



- This is well-described by a very simple differential equation called the *leaky integrator equation*
- Because every system obeying this equation can be seen as a model for this, one can use a more concrete example to illustrate the derivation and system behaviour...

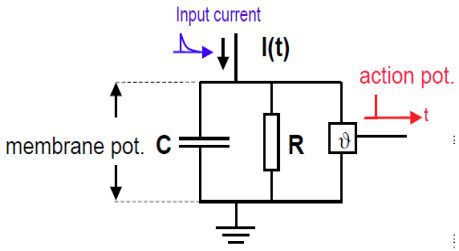
„Filling a bath, when the plughole is open“

Flow in



- **u(t)** = volume of water at time t
- **I(t)** = rate of flow in (volume/time)

Flow out

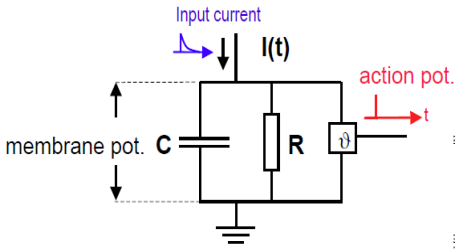


- Now more physics know-how is needed

$$\text{flow out} \propto \text{pressure} \propto \text{depth} \propto \text{volume}$$

- flow out = **k*u(t)**
- **k** constant scalar parameter
- **k>0**: water flowin out (not in)

Evaluate change rate

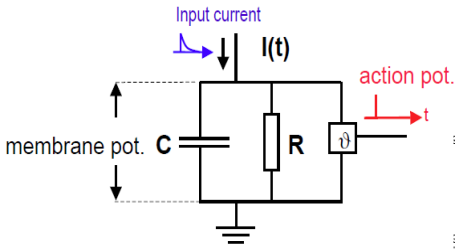


$$\frac{du}{dt} = \text{flow in} - \text{flow out} = I(t) - k \cdot u(t)$$

Now we know:

- Dynamical system (evolves around **t**)
- Differential equation (relates **du** to **u**)
- Linear in **u**
- First order (highest derivative is **du**)
- Constant coefficients (**1, K**)

Going back to Hodgkin-Huxley

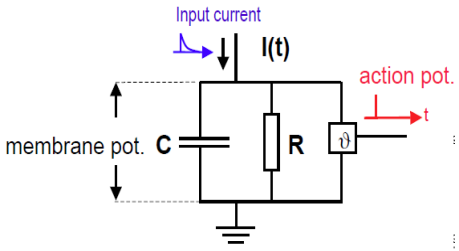


$$\frac{du}{dt} = I(t) - k \cdot u(t) \qquad \hookleftarrow \cdot \frac{1}{C}$$

$$\frac{du}{dt} = \frac{1}{C} I(t) - \frac{k}{C} \cdot u(t) \qquad \hookleftarrow \; k = \frac{1}{R}$$

$$\frac{du}{dt} = \frac{1}{C} I(t) - \frac{1}{RC} \cdot u(t) \qquad \square$$

Exact integrator



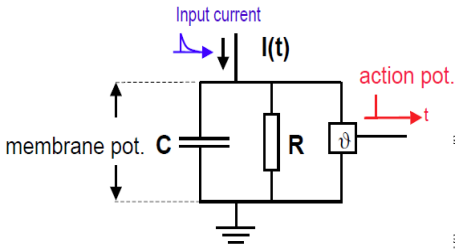
- Special case: **k = 0** (i.e. no leakage)

$$\frac{du}{dt} = I(t) \quad \xrightarrow{\text{FTC}} \quad u(t) = \int_0^t I(\tau)\,d\tau + u_0$$

$$u(t) = u_0 + \int_0^t I_{\max}\,d\tau = u_0 + \left[I_{\max}\right]_0^t$$

$$= u_0 + \left[I_{\max} \cdot t - I_{\max} \cdot 0\right] = u_0 + I_{\max} \cdot t$$

Sanity checks

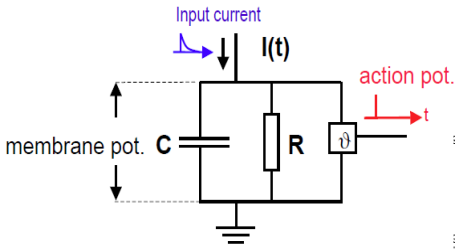


- When will the bath be full?

$$u_0 + I_{\max} \cdot t = u_{\max} \ ?$$

$$t = t_{full} = \frac{u_{\max} - u_0}{I_{\max}}$$

Transient behaviour

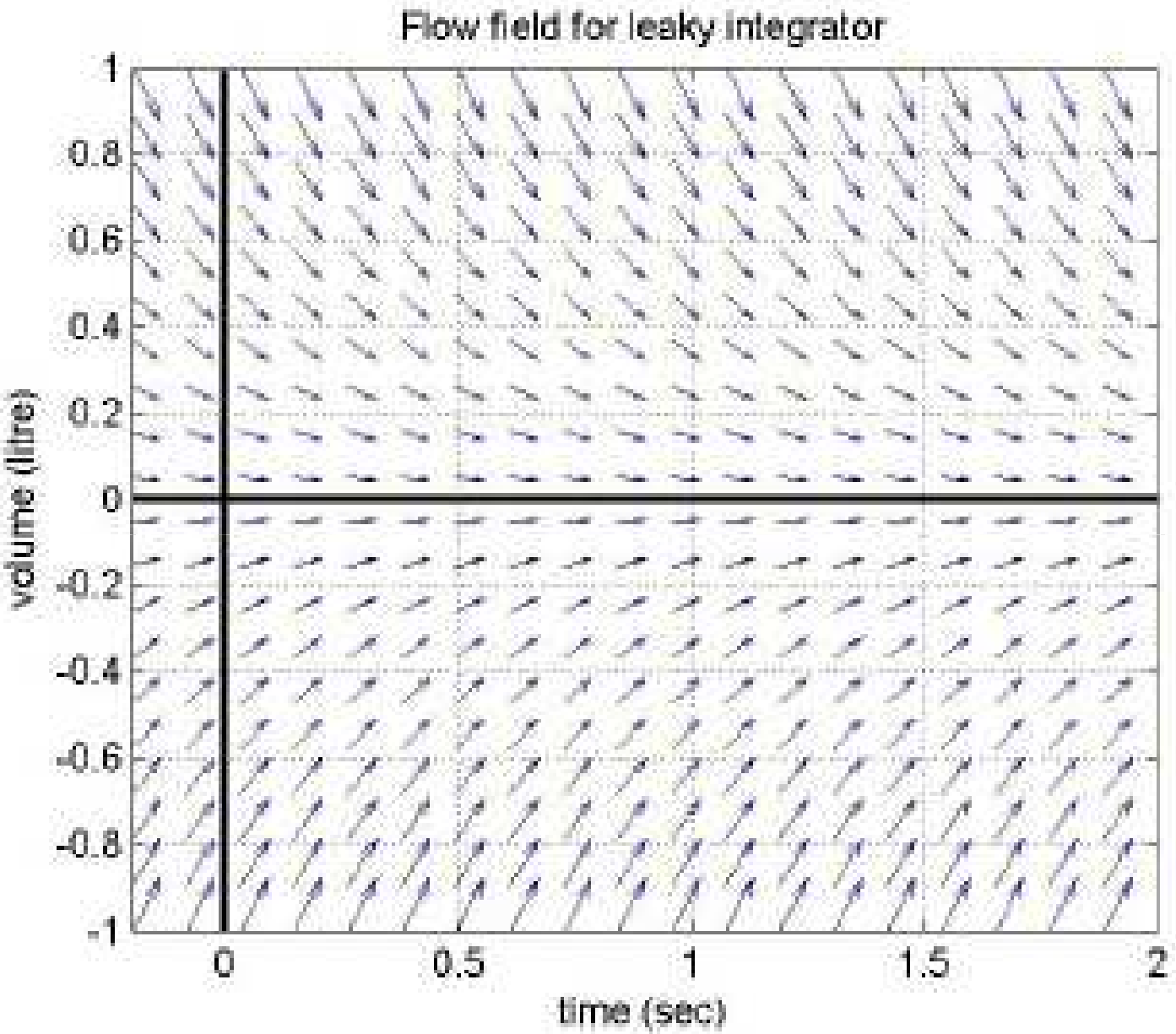


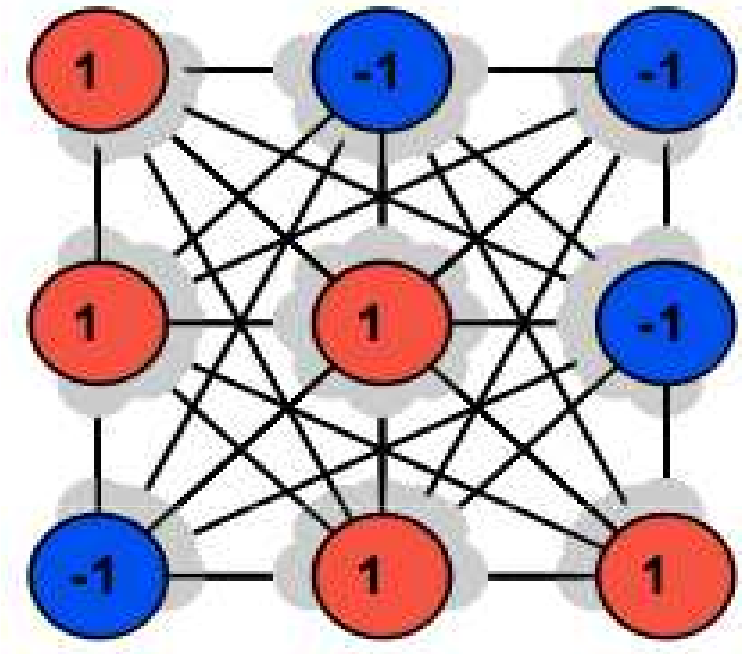
- Special case: setting **I(t) = 0**

$$\frac{du}{dt} = -k \cdot u(t) \quad \xrightarrow{\text{Intuition}} \quad u(t) = A \cdot \exp^{-kt}$$

General solution

$$u(t) = A \cdot \exp^{-kt} + \frac{1}{k} I(t)$$


Flow field for leaky integrator
volume (litre)
time (sec)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.5
1
1.5
2



Next week -> simulating with simbrain